MODELS OF CONTINUOUS MEDIA WITH NONHOLONOMIC EQUATIONS OF STATE

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1. In [1] the suggestion was made to describe the mechanical behavior of materials with complex properties by means of nonholonomic models of continuous media. Such models may be introduced in the following manner. Consider a medium in which the thermodynamic parameters of state are

$$\varepsilon_{ij}, \quad T, \quad \sigma^{ij}, \quad \chi \qquad \left(\varepsilon_{ij} = \frac{1}{2}(g_{ij}^{\wedge} - g_{ij}^{\circ}),\ \sigma^{ij} = \frac{p^{ij}}{\rho}\right) \tag{1.1}$$

Here ε_{ij} are the components of the tensor of finite deformation in a Lagrangian system of coordinates ξ^i; g_{ij}° and $g_{ij}^{\wedge}$ are the components of the metric tensor G of the initial and the deformed space, respectively; T is the temperature, p^{ij} are the contravariant components of the stress tensor in the system ξ^i, ρ is the density, and χ is some supplementary variable parameter.

The introduction of stresses as state parameters is connected with the fact that in media which are to be considered, the stresses are not functions of other parameters of system (1.1). We also emphasize, that we plan to use precisely the contravariant components of the generalized stress tensor σ^{ij}. This leads to a greater simplicity in theoretical considerations.

The explicit introduction of one or several supplementary parameters of the type χ is necessary in media with complex properties. These parameters are related to the quantitative description of internal physico-chemical processes essential for mechanics.

In addition to the system of parameters (1.1), which are variable in general, the physical properties of the medium may be determined also by a system of physical constants – scalars, vectors or tensors. The explicit enumeration of such physical constants is necessary for the construction of special models of continuous media.

Let us deduce now the consequences obtainable for media of type (1.1) from the laws of thermodynamics.

Let us designate by S and F the entropy and the free energy of an element of uniform mass, and let us write down the first and second law of thermodynamics in the form

$$dF - \sigma^{kl} d\varepsilon_{kl} + S dT = -dQ' \tag{1.2}$$

$$T dS = dQ + dQ' \tag{1.3}$$

where dQ and $\sigma^{kl} d\varepsilon_{kl}$ is the external influx of heat and the elementary work of internal surface forces calculated per unit of mass, dQ' is non-compensated heat; in irreversible processes $dQ' \geqslant 0$.

Let us assume that the irreversible processes in the media considered are related to changes in the parameter χ. Then as the simplest hypothesis we may set

$$dQ' = \varkappa \frac{d\chi}{dt} d\chi, \qquad \varkappa \geqslant 0 \tag{1.4}$$

where κ is a scalar function of parameters (1.1).

Equation (1.2), taking into account (1.1) and (1.4) may be rewritten in the form

$$\frac{\partial F}{\partial \chi} d\chi + \frac{\partial F}{\partial \sigma^{ij}} d\sigma^{ij} + \left(\frac{\partial F}{\partial \varepsilon_{kl}} - \sigma^{kl}\right) d\varepsilon_{kl} + \left(\frac{\partial F}{\partial T} + S\right) dT = -\varkappa \frac{d\chi}{dt} d\chi \tag{1.5}$$

In Equation (1.5) the differentials $d\varepsilon_{kl}$ and $d\sigma^{ij}$ are taken for a given element in the corresponding system of Lagrangian coordinates. In this case, as is known, $d\varepsilon_{kl}/dt$ are the components of the rate of deformation tensor. The quantities $d\sigma^{ij}/dt$ are conveniently taken as the components of the stress rate tensor [2].

If in addition to the assumption that the parameters (1.1) are independent one assumes that the quantities $d\chi$, $d\varepsilon_{kl}$, dT and $d\sigma^{ij}$ are also independent, then we obtain from (1.5), in particular, a system of finite equations

$$\frac{\partial F}{\partial \sigma^{ij}} = 0$$

which will permit to reduce the number of independent determining parameters in (1.1); therefore, the conservation of assumption (1.1) may be related to the presence of nonholonomic relations

$$d\sigma^{ij} = A^{ij} d\chi + B^{ij} dT + A^{ijkl} d\varepsilon_{kl} \tag{1.6}$$

where B^{ij} and A^{ijkl} are functions of the system of determining parameters (1.1), but A^{ij} may be considered as depending also on the derivatives with respect to time and on coordinates of the parameters of state. Relations (1.5) and (1.6) lead to the following equations (the increments $d\chi$, dT and $d\varepsilon_{kl}$ are assumed to be linearly independent):

$$\frac{\partial F}{\partial \sigma^{kl}} A^{klij} + \frac{\partial F}{\partial \varepsilon_{ij}} - \sigma^{ij} = 0 \qquad (i, j = 1, 2, 3) \tag{1.7}$$

$$\frac{\partial F}{\partial \sigma^{ij}} B^{ij} + \frac{\partial F}{\partial T} + S = 0 \tag{1.8}$$

$$\frac{\partial F}{\partial \chi} + \frac{\partial F}{\partial \sigma^{ij}} A^{ij} = -\varkappa \frac{d\chi}{dt} \tag{1.9}$$

Equations (1.7) and (1.8) may be considered as a generalization of the equations of state of the usual theory of elasticity. They either give the limitations on the coefficients A^{ijkl} and B^{ij}, if the free energy F and entropy S are specified, or they permit F and S to be found for prescribed A^{ijkl} and B^{ij}, in the latter case these coefficients are also subjected to limitations expressed by conditions of compatibility of equations of the system (1.7) and (1.8). Relation (1.9) is considered as the kinetic equation for the determination of the parameter χ.

Relations (1.3), (1.6) and (1.9) together with the dynamic equations and the equation of continuity may form a closed system of equations, determining a model of a continuous medium for a wide class of mechanical, thermal and physical processes, if the quantities A^{ij}, B^{ij}, A^{ijkl} and F are fixed, such that the relations (1.7) are satisfied identically; the entropy is then calculated from (1.8).

Let us show that within the framework of the suggested theory bodies with creep and relaxation may be described, as well as those with properties of hyperelasticity. Let us consider first several purely illustrative examples of such media, not connecting them with results of specific experiments on some particular materials.

2. Let us consider the example of a body with creep. Let us assume that the tensor A^{ijkl} does not depend on ε_{ij} and σ^{ij}, just as in the derivation of Hooke's law. Then we have for an isotropic body

$$A^{ijkl} = \lambda g^{\circ ij} g^{\circ kl} + \mu (g^{\circ ik} g^{\circ jl} + g^{\circ il} g^{\circ jk}) \tag{2.1}$$

where λ and μ are scalars, which may be considered as functions of T and

χ. Analogous formulas which have a more complex structure may be written down in the case when among the number of physical constants determining the medium there are vectors or tensors, i.e. for an anisotropic medium.

Let us assume further, that for χ = const the thermal effects are the same as in the usual linear thermoelasticity. Then

$$B^{ij} = \left[\frac{\partial\lambda}{\partial T} I_1(\varepsilon) - \alpha\right] g^{\circ ij} + 2\frac{\partial\mu}{\partial T}\, \varepsilon_{kl} g^{\circ ki} g^{\circ lj} \tag{2.2}$$

where $\alpha(T, \chi)$ is the coefficient of thermal expansion, $I_1(\varepsilon)$ is the first invariant of the tensor of deformation ($I_1(\varepsilon) = g^{\circ ij}\varepsilon_{ij}$)).

For such a selection of A^{ijkl} and B^{ij} relations (1.7) and (1.8) are satisfied identically if we set

$$F = \sigma^{ij}\varepsilon_{ij} - \mu I_2(\varepsilon) - \frac{\lambda}{2} I_1^2(\varepsilon) + f(T, \chi) \tag{2.3}$$

$$S = -\frac{1}{2}\left[\frac{\partial\lambda}{\partial T} I_1^2(\varepsilon) + 2\frac{\partial\mu}{\partial T} I_2(\varepsilon)\right] + \alpha I_1(\varepsilon) - \frac{\partial f}{\partial T}, \qquad (I_2(\varepsilon) = g^{\circ ij} g^{\circ kl}\varepsilon_{ik}\varepsilon_{jl}) \tag{2.4}$$

It is easily verified that for χ = const (2.3) becomes the expression for the free energy of a linear elastic body. We set further

$$-A^{ij} = \zeta I_1(\sigma) g^{\circ ij} + 2\eta\sigma^{ij} \tag{2.5}$$

$$-\varkappa = \frac{\varkappa_0}{\sqrt{I_2(\sigma)}}\left(\frac{\partial F}{\partial\chi} + A^{ij}\varepsilon_{ij}\right) \tag{2.6}$$

Here $I_1(\sigma) = g_{ij}^{\circ}\sigma^{ij}$, $I_2(\sigma) = g_{ij}^{\circ} g_{kl}^{\circ}\sigma^{ik}\sigma^{jl}$; ζ and η are scalar functions, related to the first and second viscosity of the medium; $\kappa_0 = \kappa_0(\chi, T)$.

Let us assume further, that λ, μ, α, ζ, η, κ_0 do not depend on χ, while the function $f(T, \chi)$, entering the expression for the free energy, may be represented in the formula $f_1(T) + f_2(\chi)$, whereby we can assume that $f_2(\chi) = k\chi$; the latter may always be achieved by introducing in place of χ a new parameter $\chi_1 = k^{-1} f_2(\chi)$.

For such a selection of coefficients the kinetic equation for χ takes on the form

$$\frac{d\chi}{dt} = \frac{\sqrt{I_2(\sigma)}}{\varkappa_0(T, \chi)} \tag{2.7}$$

From Equation (2.7) it is seen that under stationary conditions, i.e. for σ^{ij} = const and T = const, the parameter χ is a function of time t. In the available theories of creep for infinitely small deformations the time t is always used as the parameter χ. It is obvious, however, that under the conditions of variable temperature or variable loading this is not satisfactory. In this case the parameter χ should be used which is not a simple function of time as is reflected in the Equation (2.7). Equations (1.6) may now be written in the form

$$\frac{d\sigma^{ij}}{dt} = -\frac{\sqrt{I_2(\sigma)}}{\varkappa_0}\left[\zeta I_1(\sigma)\, g^{\circ ij} + 2\eta\sigma^{ij}\right] + \left(\frac{d\sigma'^{ij}}{dt}\right)_{\chi=\text{const}} \tag{2.8}$$

where

$$\sigma'^{ij} = \lambda I_1(\varepsilon)\, g^{\circ ij} + 2\mu\varepsilon^{ij} - g^{\circ ij}\int_{T_*}^{T} \alpha\, dT$$

Formulas (2.3), (2.7) and (2.8) define completely a certain model of a continuous medium. The arbitrariness contained in them in the form of undetermined functions α, λ, μ, ζ, η, κ_0, f_1, k is removed by means of results of very simple experiments in the case when these equations are applied for the description of some specific material.

Let us investigate the behavior of this medium under various conditions with the example of uniaxial extension - compression.

Let us direct the axes of a Lagrangian system of coordinates ξ^i along the principal axes of deformation, whereby the specimen is stretched by the force acting along the axes ξ^1. Let the system of coordinates at the initial instant be a Cartesian one

$$g^{\circ ij} = g_{ij}{}^{\circ} = \delta_j^{\,i}.$$

During the whole process of deformation only the components σ^{11}, ε_{11}, $\varepsilon_{22} = \varepsilon_{33}$, are different from zero, and in the system (2.8) only two equations are essential

$$E\frac{d\varepsilon_{11}}{dt} = \frac{d\sigma^{11}}{dt} + \beta(\sigma^{11})^2 - \left\{(1-2\nu)\left[I_1(\varepsilon)\frac{d\lambda}{dT} - \alpha\right] + 2\frac{d\mu}{dT}(\varepsilon_{11} - 2\nu\varepsilon_{22})\right\}\frac{dT}{dt} \tag{2.9}$$

$$E\frac{d\varepsilon_{22}}{dt} = -\nu\frac{d\sigma^{11}}{dt} + \left[\beta - \frac{2\eta(1+\nu)}{\varkappa_0}\right](\sigma^{11})^2 - \left\{(1-2\nu)\left[I_1(\varepsilon)\frac{d\lambda}{dT} - \alpha\right] - 2\frac{d\mu}{dT}\left[\nu\varepsilon_{11} - (1-\nu)\varepsilon_{22}\right]\right\}\frac{dT}{dt}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{1}{2}\,\frac{\lambda}{\lambda + \mu}, \qquad \beta = \frac{\zeta(1-2\nu) + 2\eta}{\varkappa_0}$$

The solutions of this system are of the form:

a) in the case of isothermal stress relaxation

$$T = T_0, \qquad \varepsilon_{11} = \varepsilon_{11}^{*}$$

$$\sigma^{11} = \frac{\sigma^{\circ 11}}{1 + \beta\sigma^{\circ 11}t}, \qquad \varepsilon_{22} = \varepsilon_{22}{}^{\circ} - \frac{\zeta}{\varkappa_0}\,\frac{(1+\nu)(1-2\nu)}{E\beta}\,(\sigma^{11} - \sigma^{\circ 11})$$

$$\sigma^{\circ 11} = \sigma^{11}, \qquad \varepsilon_{22}{}^{\circ} = \varepsilon_{22} \quad \text{for } t = 0$$

b) in the case of isothermal creep for fixed σ^{11}

$$T = T_0, \qquad \sigma^{11} = \sigma^{\circ 11}$$

$$\varepsilon_{11} = \varepsilon_{11}{}^{\circ} + \frac{\beta}{E}(\sigma^{\circ 11})^2 t, \qquad \varepsilon_{22} = \varepsilon_{22}{}^{\circ} + \left(\frac{\beta}{E} - \frac{2\eta}{\varkappa_0}\frac{1+\nu}{E}\right)(\sigma^{\circ 11})^2 t$$

$$\varepsilon_{11}{}^{\circ} = \varepsilon_{11}, \qquad \varepsilon_{22}{}^{\circ} = \varepsilon_{22} \quad \text{for } t = 0$$

c) in the case of isothermal deformation with constant velocity V

$$T = T_0, \qquad \varepsilon_{11} = Vt$$

$$\sigma^{11} = a_0 \sqrt{V} \tanh \frac{\delta_0}{\sqrt{V}} \varepsilon_{11}, \qquad \varepsilon_{22} = \left[1 - \frac{2\eta}{\varkappa_0}\frac{1+\nu}{\beta}\right]\varepsilon_{11} - \frac{\zeta}{\varkappa_0}\frac{(1+\nu)(1-2\nu)}{E\beta}\sigma^{11}$$

$$a_0 = (E/\beta)^{1/2}, \qquad \delta_0 = \sqrt{E\beta}; \qquad \sigma^{11} = 0, \quad \varepsilon_{22} = 0 \quad \text{for } t = 0$$

The solutions obtained make it possible to fix the parameters entering into the definition of the medium by comparison with results of very simple experiments on specific materials. Thus, for instance, if for fixed ε_{11} the quantity of deformation ε_{22} does not change either, the parameter ζ is equal to 0.

Let us now consider processes connected with temperature changes during deformation. In this case the energy equation must be used

$$dU - \sigma^{ij} d\varepsilon_{ij} = dQ \tag{2.10}$$

where dU is the increase in the internal energy of the medium, and dQ is the external influx of heat.

Let us assume for simplicity that λ, μ, α, k, κ_0, ζ, η do not depend on T. Then

$$U = F - TS = \sigma^{ij}\varepsilon_{ij} - \frac{\lambda}{2} I_1{}^2(\varepsilon) - \mu I_2(\varepsilon) + \alpha T I_1(\varepsilon) + \left(f_1 - T\frac{\partial f_1}{\partial T}\right) + k\chi \tag{2.11}$$

It is easily shown that $c = \partial\varphi/\partial T$, where $\varphi = f_1 - T\partial f_1/\partial T$ is the heat capacity of the medium for constant deformations. The quantity c is assumed to be constant.

For the one-dimensional processes considered the energy equation takes on the form

$$-\frac{\zeta + 2\eta}{\varkappa_0}(\sigma^{11})^2\varepsilon_{11} - 2\frac{\zeta}{\varkappa_0}(\sigma^{11})^2\varepsilon_{22} + k\frac{\sigma^{11}}{\varkappa_0} + c\frac{dT}{dt} + \alpha T\left(\frac{d\varepsilon_{11}}{dt} + 2\frac{d\varepsilon_{22}}{dt}\right) = \frac{dQ}{dt} \tag{2.12}$$

Let us apply equation (2.12) for the study of the process of adiabatic creep. This process is described by the system of equation

$$E\frac{d\varepsilon_{11}}{dt}=\frac{2\eta}{\varkappa_0}(\sigma^{\circ 11})^2+\frac{dT}{dt}\alpha(1-2\nu), \quad E\frac{d\varepsilon_{22}}{dt}=-\frac{2\eta\nu}{\varkappa_0}(\sigma^{\circ 11})^2+\frac{dT}{dt}\alpha(1-2\nu) \quad (2.13)$$

$$\alpha T\left(\frac{d\varepsilon_{11}}{dt}+2\frac{d\varepsilon_{22}}{dt}\right)+c\frac{dT}{dt}=\frac{2\eta}{\varkappa_0}(\sigma^{\circ 11})^2\varepsilon_{11}-\frac{k}{\varkappa_0}\sigma^{\circ 11}$$

Here, for simplicity, we set $\zeta = 0$. The solution of this system is given by the formulas

$$\varepsilon_{11}=\varepsilon_{11}^{\circ}+\frac{\alpha}{3K}(T-T_0)+\frac{2\eta}{\varkappa_0}(\sigma^{\circ 11})^2 t \qquad \left(3K=\frac{E}{1-2\nu}\right)$$

$$\varepsilon_{22}=\varepsilon_{22}^{\circ}+\frac{\alpha}{3K}(T-T_0)-\frac{2\eta\nu}{\varkappa_0}\ \sigma^{\circ 11})^2 t$$

$$\frac{[c+\alpha^2 TK^3]^2}{\alpha^2K^3}-\frac{E\,[2\eta(\sigma^{\circ 11})^2 t/\varkappa_0 E+(\varepsilon_{11}^{\circ}-\alpha T_0/3)+k/2\eta\sigma^{\circ 11}]^2}{1}=M_0$$

$$\varepsilon_{11}^{\circ}=\varepsilon_{11}, \qquad \varepsilon_{22}^{\circ}=\varepsilon_{22}, \qquad T_0=T \quad \text{for } t=0$$

$$M_0=\frac{[c+\alpha^2 T_0 K^3]^2}{\alpha^2 K^3}-\frac{E\,[(\varepsilon_{11}^{\circ}-\alpha T_0/3)+k/2\eta\sigma^{\circ 11}]^2}{1}$$

For $\zeta \neq 0$ the solution is given by the same formulas with different constants.

Thus the temperature in the process of adiabatic creep increases with time. The deformations thereby are increasing faster than in the case of isothermal creep. Experiments with variable temperature permit the values α and c to be determined for a given medium.

3. Let us consider the example of a hyperelastic material. Let it be required to describe the behavior of the material in which large reversible deformations can occur which increase in time. The model of such a medium may be constructed if we consider that $\kappa = 0$ (there is no dissipation connected with deformation), k is a quantity which is essentially different from 0 (the internal energy necessarily depends on χ) and the A^{ij} are such functions of $d\chi/dt$ that the nonholonomic relations (1.6) have a reversible character, i.e. at the changes of the sign of the differentials $d\chi$, dT and $d\varepsilon_{ij}$ the quantities $d\sigma^{ij}$ also change sign. Consider, for instance,

$$A^{ij}=\Lambda\left(\frac{d\chi}{dt}\right)^2\frac{\varepsilon^{ij}}{I_2(\varepsilon)}, \qquad \Lambda=\frac{-k}{[aI_2(\varepsilon)-bI_2(\sigma)]^2} \tag{3.1}$$

Then the kinetic equation for χ takes on the form

$$\frac{d\chi}{dt}=-[aI_2(\varepsilon)-bI_2(\sigma)] \tag{3.2}$$

with the condition that the free energy is again given by the form (2.3).

If A^{ijkl} and B^{ij} are selected the same as in the preceding example, then the relations (1.6) may be written down in the following form:

$$\frac{d\sigma^{ij}}{dt} = \frac{k}{I_2(\varepsilon)}\left[aI_2(\varepsilon) - bI_2(\sigma)\right]\varepsilon^{ij} + \left(\frac{d\sigma'^{ij}}{dt}\right)_{\chi=\text{const}} \tag{3.3}$$

Let us consider the process of isothermal uniform loading of such a medium for conditions $I_1(\varepsilon) = 0$. They are described by the equation

$$\frac{d\sigma^{11}}{dt} = \frac{3k\left[aI_2(\varepsilon) - bI_2(\sigma)\right]}{2I_2(\varepsilon)}\varepsilon^{11} + 3\mu\frac{d\varepsilon^{11}}{dt} \tag{3.4}$$

whereby $I_2(\varepsilon) = 3\varepsilon_{11}{}^2/2$, $I_2(\sigma) = (\sigma^{11})^2$. It is obvious that the state $\varepsilon_{11} = \varepsilon_{11}{}^*$, $\sigma^{11} = \sigma^{*11}$ with the condition

$$\frac{3}{2}a(\varepsilon_{11}{}^*)^2 = b(\sigma^{*11})^2 \tag{3.5}$$

is an equilibrium state. It satisfies the Equations (3.2) and (3.4); thereby $\chi = \chi^* = \text{const}$. Let us show that in the processes of deformations with $\sigma^{11} = \text{const}$ and in processes of variable stress with $\varepsilon_{11} = \text{const}$ the material considered approaches an equilibrium state, determined by condition (3.5).

Let $\sigma^{11} = \sigma^{\circ 11} = \text{const}$. Then from (3.4) it follows

$$\frac{3}{2}a\varepsilon_{11}{}^2 - b(\sigma^{\circ 11})^2 = \left[\frac{3}{2}a(\varepsilon_{11}{}^\circ)^2 - b(\sigma^{\circ 11})^2\right]\exp\left(-\frac{ka}{\mu}t\right)$$

$$\varepsilon_{11}{}^\circ = \varepsilon_{11} \qquad \text{for } t = 0$$

Thus, $\varepsilon_{11} \to \sqrt{(2b/3a)}\sigma^{\circ 11} = \varepsilon_{11}{}^*$ as $t \to \infty$, whereby, if the initial deformation $\varepsilon_{11}{}^\circ$ is smaller than $\varepsilon_{11}{}^*$, corresponding to the given stress $\sigma^{\circ 11}$, then a process of creep takes place, i.e. the deformation increases until the equilibrium state is reached. In the opposite case a process of restitution takes place; for fixed stress $\sigma^{\circ 11}$ the deformation of the specimen decreases until the equilibrium state is reached. If $\sigma^{\circ 11} = 0$, then also $\varepsilon_{11}{}^* = 0$ and, consequently, when the stress is relieved complete resitution takes place.

The change of stresses for fixed $\varepsilon_{11} = \varepsilon_{11}{}^\circ$ takes place according to the law

$$\sqrt{\frac{2}{3}\frac{b}{a}}\frac{\sigma^{11}}{\varepsilon_{11}{}^\circ} = \frac{1 - C\exp(-\sqrt{3ab/2}\,t)}{1 + C\exp(-\sqrt{3ab/2}\,t)}$$

where C is related to $\sigma^{\circ 11}$, the initial value of σ^{11}. Consequently, $\sigma^{11} \to \sqrt{(3a/2b)}\varepsilon_{11}{}^\circ = \sigma^{*11}$ as $t \to \infty$. If $\sigma^{\circ 11} > \sigma^{*11}$, then the process of stress relaxation takes place; for $\sigma^{\circ 11} < \sigma^{*11}$ stresses increase for fixed deformation.

4. The examples considered were of qualitative and illustrative character. In a series of cases, however, a quantitative description of real materials may be given in the framework of the considered system on nonholonomic models.

Let us present now a simple model which describes with sufficient accuracy the results of the experiments by Bergen, Messersmith and Rivlin [3] on relaxation of stresses in certain filled high-polymers.

In these experiments five materials were used: vulcanized, heavily filled synthetic rubber and four polyvinyl chloride compositions containing various amounts of inorganic fillers (from 0% to 50% by volume).

The specimens from these materials in the shape of straight tubes of constant circular cross-section with internal and external radii a and b were subjected simultaneously to simple extension and torsion in the course of a small interval of time $0 \leqslant t \leqslant \tau$ with constant temperature. The deformations were fixed subsequently and for $t \gg \tau$ the forces were measured which were necessary to maintain these deformations.

The results of experiments were used to construct a mathematical model which would permit the description of the materials considered during processes of relaxation. The authors show that in the experiments one may assume with sufficient accuracy that the materials are incompressible and that in processes of relaxation the following relations are satisfied

$$p^{ij} = \theta\varepsilon_{ij} - p\delta_{ij} \tag{4.1}$$

where p^{ij} are the components of the stress tensor, p is the hydrostatic pressure, ε_{ij} are the components of the small strain tensor, θ is the scalar function of time t and two strain invariants ($I_1(\varepsilon) = 0$ by virtue of the condition of incompressibility by small deformations).

The scalar function θ was found by the authors for all five materials considered. It is clear that the Equations (4.1) in which time t appears explicitly, do not serve as a mathematical model of either material. These equations do preclude the possibility of not only studying the behavior of materials in processes connected with the change of the strain tensor or the temperature, but also regarding the initial stage of the process of relaxation by ε_{ij} = const. At the same time a whole class of models with relations of the type (1.6) may be suggested which describe the behavior of the material in any process and by which the final stage of relaxation for fixed small strains obeys the Equations (4.1). Let, for instance,

$$A^{ij}=\varepsilon^{ij}+\frac{\alpha}{\chi}\sigma^{ij},\ \alpha>0, \tag{4.2}$$

where $\alpha = \alpha(\varepsilon_{ij}, T)$ is a function which must be determined from experiments. The function κ will be determined in such a way, that the kinetic equation for χ is of the form of

$$\frac{d\chi}{dt}=-M(\varepsilon_{ij}, T)\chi^{\beta},\qquad \beta>1 \tag{4.3}$$

M and $\beta = \beta(T)$ are also determined from experiments.

Then for an arbitrary selection of coefficients B^{ij} and A^{ijkl} in relations (1.6) the isothermal relaxation of stresses for fixed strains obeys the following laws:

$$d\sigma^{ij}=\left(\varepsilon^{ij}+\frac{\alpha}{\chi}\sigma^{ij}\right)d\chi$$

or

$$d(\sigma^{ij}-\sigma g^{\circ ij})=\left[\varepsilon^{ij}+\frac{\alpha}{\chi}(\sigma^{ij}-\sigma g^{\circ ij})\right]d\chi \tag{4.4}$$

In these formulas $g^{\circ ij}$ are the components of G in undeformed space, $\sigma = \sigma^{ij}g_{ij}^{\circ}/3$, the strains are small and the material is incompressible: $I_1(\varepsilon) = 0$.

The solution of system (4.3) and (4.4) is given by the equations

$$\chi=[M(\beta-1)t+C_0]^{-\frac{1}{\beta-1}} \tag{4.5}$$

$$\sigma^{ij}-\sigma g^{\circ ij}=\frac{\chi}{\alpha-1}\varepsilon^{ij}+\left[\sigma^{\circ ij}-\sigma^{\circ}g^{\circ ij}-\frac{\chi_0}{\alpha-1}\varepsilon^{ij}\right]\left(\frac{\chi}{\chi_0}\right)^{\alpha} \tag{4.6}$$

where

$$\sigma^{\circ ij}=\sigma^{ij},\qquad \sigma^{\circ}=\sigma,\qquad \chi_0=\chi\qquad \text{for } t=0\qquad C_0=\frac{1}{\chi_0^{\beta-1}}$$

For large times $t(t \gg \tau)$ these formulas become

$$\chi=[M(\beta-1)t]^{-\frac{1}{\beta-1}} \tag{4.7}$$

$$\sigma^{ij}-\sigma g^{\circ ij}=\frac{\chi}{\alpha-1}\varepsilon^{ij}=(\alpha-1)^{-1}[M(\beta-1)t]^{-\frac{1}{\beta-1}}\varepsilon^{ij} \tag{4.8}$$

The quantity τ is set equal to the largest of t_1 and t_{ij}, whereby

$$t_1 = \frac{C_0}{M(\beta-1)}, \qquad t_{ij} = \frac{C_0{}^{\alpha/\alpha-1}[(\alpha-1)(\sigma^{oij}-\sigma^o g^{oij})]^{(\beta-1)/(\alpha-1)}}{M(\beta-1)(\varepsilon^{ij})^{(\beta-1)/(\alpha-1)}} - \frac{C_0}{M(\beta-1)}$$

$$(i, j = 1, 2, 3)$$

Thus the final stage of processes of stressed relaxation in the media of the type considered is described by formulas which coincide in the construction with those of the authors [3]. Indeed:

1) the stresses do not depend on their initial value; this means, that the material "forgets" the history of initial deformations;

2) the dependence of stresses on time and deformation is given in the form

$$\sigma^{ij} - \sigma g^{oij} = \varphi_1(t)\,\varphi_2(\varepsilon_{ij}, T_0)\,\varepsilon^{ij}$$

where φ_1 does not depend on deformation, nor φ_2 - on time;

3) the function

$$\varphi_1 = t^{-\frac{1}{\beta-1}}$$

is a decreasing function of time t by $\beta > 1$. On the basis of experimental results given in form of tables in [3], the value of β may be determined for a temperature $T = 21 \pm 1^\circ C$, which corresponds to the conditions of the experiments and also the dependence of the function

$$(\alpha-1)^{-1}[M(\beta-1)]^{-\frac{1}{\beta-1}}$$

on deformation for the same temperature. Thus, for instance, for specimens made of polyvinyl chloride composition No. 1 we may assume

$$\beta - 1 = 7.64, \qquad \rho(\alpha-1)^{-1}[M(\beta-1)]^{-\frac{1}{\beta-1}} = A$$

where A is a constant for given temperature and is equal to 1.85×10^3 lb/in^2; for specimens of polyvinyl chloride composition No. 3

$$\beta - 1 = 6.38, \qquad \rho(\alpha-1)^{-1}[M(\beta-1)]^{-\frac{1}{\beta-1}} = A - B \log J_1$$

where $J_1 = 2(\varepsilon_{ij}\varepsilon_{ji} - \varepsilon_{ii}\varepsilon_{jj})$, $A = 0.89 \times 10^3$ lb/in^2, $B = 0.86 \times 10^3$ lb/in^2 for the same temperature of the experiments.

However, the tests conducted in [3], are entirely insufficient for a complete determination of the model. They do not permit to select χ_0, because the initial stage of the relaxation was not investigated; α and M cannot be determined from them separately, the dependence of all the functions of temperature, the coefficients B^{ij}, A^{ijkl}, κ, cannot be selected in a suitable way, nor can the form of the free energy F. For a complete determination of the model tests must be conducted in particular on isothermal creep and on deformation under conditions of variable temperature.

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